Introduction to Dynamical Systems

Solutions Problem Set 11

Exercise 1. Show that the set of *non-diophantine* irrational numbers in $\mathbb{R} \setminus \mathbb{Q}$ is of Lebesgue measure 0.

Solution. Notice that we may restrict ourselves to numbers β the interval $[0, 1]$, as any non-diophantine number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ only differs from β by an integer. Now notice that for each $d \in \mathbb{N}$, we have (p_d, q_d) such that

$$\left|\beta - \frac{p_d}{q_d}\right| < c|q_d|^{-(d+1)}.$$

This means that if we define

$$A_q^d = \left[0, \frac{c}{q^{d+1}}\right) \cup \left(1 - \frac{c}{q^{d+1}}, 1\right] \cup \bigcup_{i=1}^{q-1} \left(\frac{i}{q} - \frac{c}{q^{d+1}}, \frac{i}{q} + \frac{c}{q^{d+1}}\right),$$

then β belongs to $A_{q^d}^d$ for all d, so all non-diophantine numbers in $[0, 1]$ let us name this set N must be in an infinite number of the sets $A_{q^d}^d$, meaning that if

$$B_n = \bigcup_{d=n}^{\infty} A_{q^d}^d, \quad \text{then} \quad N \subset \bigcap_{n=1}^{\infty} B_n.$$

We finally compute to find

$$|A_{q^d}^d| \leq 4c/q_d^{d+1}, \quad |B_n| \leq 4c,$$

and thus

$$|N| \leq \liminf_{n\to\infty} |B_n| = 0.$$

□

Exercise 2. Recall (or look up) the three lines lemma of Hadamard. Using it, show the inequality

$$\sum_{k\geq 0} k^d \, |\zeta|^k \leq C(d)(1 - |\zeta|)^{-(d+1)}$$

provided $|\zeta| < 1$, and $d \geq 0$. Can you estimate $C(d)$?

Solution. First of all, call $x = |\zeta| \in (0, 1)$ and notice that by setting

$$f_x(z) = (1 - x)^{z+1} \sum_{k\geq 0} k^z x^k,$$

we have that f_x is a holomorphic function on any strip of the form $[n - 1, n] \times i\mathbb{R} \subset \mathbb{C}$. Letting $z = r + is$, we come to find that

$$M(r) := \sup_s |f_x(r + is)| = \sup_s \left|\sum_{k\geq 0} k^r x^k (k^{is})\right| |1 - x|^{r+1} \left|(1 - x)^{is}\right| = f_x(r),$$

as all the terms $k^r x^k$ inside the sum are positive. We can therefore apply the Three Lines Lemma to conclude that

$$M(r) \leq M(n-1)^t M(n)^{1-t}, \quad r = t(n-1) + (1-t)n.$$

Hence we need only show the estimate for $d \in \mathbb{N}$. To that end, notice that we may define a linear operator T as

$$T(f)(z) = zf'(z),$$

in terms of which we may express f as

$$f_x(d) = (1-x)^{d+1} T^d \left(\frac{1}{1-x} \right).$$

Upon computing T^d we see that it is of the form

$$T^{d+1} \left(\frac{1}{1-x} \right) = \frac{P_d(x)}{(1-x)^{d+1}},$$

where P_d is a polynomial of degree d that satisfies the recurrence

$$P_{d+1}(x) = x(1-x)P_d'(x) + (d+1)xP_d(x), \quad P_0(x) = 1.$$

At this point it is an easy induction exercise to check that P_d always has nonnegative coefficients, and that $P_d(1) = d!$, meaning that we can estimate

$$f_x(d) = (1-x)^{d+1} \sum_{k \geq 0} k^d x^k = (1-x)^{d+1} T^{d+1} \left(\frac{1}{1-x} \right) = P^d(x) \leq P^d(1) = d!$$

We therefore conclude the desired bound, as well as the estimate

$$C(d) = \lfloor d! \rfloor^t \lceil d! \rceil^{1-t}, \quad d = t\lfloor d \rfloor + (1-t)\lceil d \rceil$$

□

Exercise 3. Given any non-polynomial holomorphic function $u(z)$ on a disc $B_r(0)$, $r > 0$, show that there is an irrational number $\alpha \in \mathbb{R}$ such that

$$w(\lambda z) - \lambda w(z) = u(z), \quad \lambda = e^{2\pi i \alpha}$$

does not have a holomorphic solution on any disc $B_\delta(0)$.

Solution. Assume u is of the form in (3.5) in Lecture10.pdf and argue similarly as in Lecture11.pdf to conclude. □